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On the surface area of an ellipsoid and related integrals of elliptic integrals

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Abstract

By considering the integral of an arbitrary field on the surface of an ellipsoid of revolution from two different perspectives, two different expressions are derived for it. One of them has a complete elliptic integral as its kernel, while the other kernel is just a radical (albeit containing one additional integration). As the expressions with the latter kernel are often more readily evaluated, a new class of integrals (and some integral relations), having a complete elliptic integral as their kernel, have been established. As the integral, defining the surface area of an arbitrary ellipsoid, can also be transformed to the previous form, its value can be established in terms of elliptic integrals.

Key words: Surface area ellipsoid; Integrals of elliptic integrals; Complete elliptic integral transform

1. Introduction

Let us consider an ellipsoid S with semi-axes a, b and c, where, without loss of generality, it is assumed that $a \geqslant b \geqslant c \geqslant 0$. It is conventional to choose a right-handed Cartesian x, y, z-coordinate frame which is aligned with the longest, middle and shortest axis respectively, such that S reads

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

A point P on S has coordinates $\boldsymbol{r} = (x, y, z)$:

$$x = a \sin\theta \cos\phi, \qquad y = b \sin\theta \sin\phi, \qquad z = c \cos\theta, \tag{1}$$

where, as usual, $\phi \in [0, 2\pi)$, the "longitude", measures the angle with respect to the positive x-axis and $\theta \in [0, \pi)$, the "co-latitude", the angle from the positive z-axis.

This is, however, by no means the only possible "natural" choice of orientation of the coordinate axes and we might as well take a coordinate frame in which the Cartesian

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coordinates (x', y', z') are taken, for instance, along the shortest, middle and longest axis, such as occurs when the initial coordinate frame is rotated along the y-axis over $\frac{1}{2}\pi$. In this frame the coordinates of P read

$$x' = c \sin\theta \cos\phi, \qquad y' = b \sin\theta \sin\phi, \qquad z' = a \cos\theta, \tag{2}$$

where the same conventions for angles ϕ and θ have been adopted, but now with respect to the x'- and z'-axes. It is obvious that this choice should leave the evaluation of any field $F(x, y, z)$, integrated over the surface of the ellipsoid S,

$$\int_S F(x, y, z)\, \mathrm{d}A \tag{3}$$

unaffected. Here $\mathrm{d}A$ denotes a surface increment of S. Yet, the different orientations of coordinate axes lead to different integral expressions, which are therefore, by the previous observation, equivalent.

In the sequel a more detailed elaboration of this observation is presented for arbitrary functions $F(x, y, z)$. Some more specific choices (Section 2) enable us to evaluate (or derive various identities among) integrals containing complete elliptic integrals as their kernel. As an application the surface area of an arbitrary ellipsoid can be obtained explicitly. In Section 3 some possible generalizations are discussed and some conclusions are drawn.

It is perhaps useful to finish this Introduction with the simpler case of a two-dimensional application of the invariance to different orientations of the coordinate frames of the integral value of a field $F(x, y)$, evaluated on the perimeter of an ellipse. Consider for that purpose, e.g., the *length* of the ellipse of semi-axes a, b, $a \geqslant b$, i.e., an evaluation of the unit field

$$F(x, y) = 1.$$

When a point P of the ellipse is given by the coordinates

$$x = a \cos\phi, \tag{4a}$$

$$y = b \sin\phi, \tag{4b}$$

an increment in length along the perimeter $\mathrm{d}s$ is given by

$$\mathrm{d}s = \sqrt{x_\phi^2 + y_\phi^2}\, \mathrm{d}\phi,$$

where a subscript denotes a derivative. Then, the total length s is obtained by integrating ϕ from 0 to 2π and one obtains

$$s = 4bE(\mathrm{i}k),$$

where $E(\kappa)$ is the complete elliptic integral of the second kind, defined as

$$E(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2\phi}\, \mathrm{d}\phi,$$

and $k^2 = (a/b)^2 - 1 \geqslant 0$. Alternatively, after rotating the coordinate frame in positive direction over $\frac{1}{2}\pi$ (while the ellipse remains fixed in the plane) one may take

$$x' = b \cos\phi, \tag{5a}$$

$$y' = a \sin\phi, \tag{5b}$$

and obtain analogously

$$s' = 4aE(\kappa),$$

where $\kappa^2 = 1 - (b/a)^2 = k^2/(1+k^2)$. The equivalence of s and s' then leads to the well-known expression for a complete elliptic integral with purely imaginary modulus

$$E(\mathrm{i}k) = \sqrt{1+k^2}\,E\left(\frac{k}{\sqrt{1+k^2}}\right), \tag{6}$$

a real quantity.

The integration of an arbitrary function $F(x, y)$ in the two frames of reference (in the second of which it reads $F(y', -x')$) establishes the general identity (using (4) and (5))

$$\int_0^{\pi/2} F(a\cos\phi, b\sin\phi)\sqrt{1+k^2\sin^2\phi}\,\mathrm{d}\phi$$
$$= \sqrt{1+k^2}\int_0^{\pi/2} F(a\sin\phi, -b\cos\phi)\sqrt{1-\frac{k^2}{1+k^2}\sin^2\phi}\,\mathrm{d}\phi,$$

which, in this two-dimensional case, is trivial as the transformation $\phi = \phi' + \frac{1}{2}\pi$ readily leads to this result. In the three-dimensional case, however, these reorientations lead to nontrivial results, which will be discussed now.

2. Evaluation of integrals on an ellipsoid

The parametrization (1) of the ellipsoid S is here written as

$$x(\theta, \phi) = aX(\theta, \phi), \qquad y(\theta, \phi) = bY(\theta, \phi), \qquad z(\theta) = cZ(\theta), \tag{7}$$

where

$$X(\theta, \phi) \equiv \sin\theta\cos\phi, \qquad Y(\theta, \phi) \equiv \sin\theta\sin\phi, \qquad Z(\theta) \equiv \cos\theta, \tag{8}$$

with $(\theta, \phi) \in [0, \pi) \times [0, 2\pi) \equiv B$. It is well known that an infinitesimal rectangular element of B corresponds to a parallelogram of S whose area $\mathrm{d}A$ is given by (e.g., [1])

$$\mathrm{d}A = |\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi|\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

With (7) and (8) the determinant reads

$$D(\theta, \phi; a, b, c) \equiv |\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi|$$
$$= \sin\theta\left(b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta\right)^{1/2}. \tag{9}$$

Now, as discussed in the Introduction, the evaluation of the integral value of any field $F(x, y, z)$

$$\int_S F(x, y, z)\,\mathrm{d}A = \int_0^{2\pi}\int_0^{\pi} F(aX(\theta, \phi), bY(\theta, \phi), cZ(\theta))D(\theta, \phi; a, b, c)\,\mathrm{d}\theta\,\mathrm{d}\phi$$

should be independent of the orientation of the coordinate frame. With, e.g., a rotation of $\frac{1}{2}\pi$ about the y-axis,

$$x' = -z, \qquad y' = y, \qquad z' = x, \tag{10}$$

recognizing the change in stretching factors in x- and z-direction due to the new orientation with respect to the fixed ellipsoid (but retaining the parametrization convention) now yields

$$x' = cX(\theta, \phi), \qquad y' = bY(\theta, \phi), \qquad z' = aZ(\theta). \tag{11}$$

The use of the same parametrization convention guarantees that

$$|\boldsymbol{r}'_\theta \times \boldsymbol{r}'_\phi| \equiv D'(\theta, \phi; a, b, c) = D(\theta, \phi; c, b, a), \tag{12}$$

having the same θ, ϕ-dependence, which is therefore suppressed in the following. Note, however, that the a and c dependencies in $D(a, b, c)$ are permuted in the rotated frame. With the use of (7) and (10)–(12) the equality

$$\int_0^{2\pi}\int_0^{\pi} F(x, y, z)D(a, b, c)\, \mathrm{d}\theta\, \mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi} F(z', y', -x')D'(a, b, c)\, \mathrm{d}\theta\, \mathrm{d}\phi$$

then reads, also suppressing the θ, ϕ-dependence in X, Y and Z:

$$\int_0^{2\pi}\int_0^{\pi} F(aX, bY, cZ)D(a, b, c)\, \mathrm{d}\theta\, \mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi} F(aZ, bY, -cX)D(c, b, a)\, \mathrm{d}\theta\, \mathrm{d}\phi. \tag{13}$$

More generally, any reorientation of x, y and z due to a (repeated) rotation of coordinate axes (which recognizes the existing symmetries of the ellipsoid), and corresponding permutation of a, b and c in $D(a, b, c)$, leads to another alias of the integral considered. Eq. (13) and its alternatives are the most general identities "derived" here. In what follows, the arbitrariness of $F(x, y, z)$ is in one way or another restricted.

Fields $F(x, y, z)$ which are antisymmetric in any one of the three parameters, of course, yield a zero integral value after evaluating it on the symmetrical ellipsoid. Restricting ourselves to fields which are symmetric in each of the parameters thus leaves six possible permutations of the three parameters x, y and z and establishes the equivalence of

$$\int_0^{2\pi}\int_0^{\pi} F(aX, bY, cZ)D(a, b, c)\, \mathrm{d}\theta\, \mathrm{d}\phi \tag{14}$$

under each permutation of the triplet (X, Y, Z) in F and the simultaneous, corresponding permutation of the triplet (a, b, c) in D. Note that the adopted restriction eliminates a consideration of asymmetric (skew) fields in the remainder.

Example (*Integral expression for the surface area of an ellipsoid*). As an illustrative example consider the unit field

$$F(x, y, z) = 1,$$

which yields an integral representation of the surface area of an ellipsoid S. With (9) and (14) this area is given as

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{\pi} D(a, b, c)\, \mathrm{d}\theta\, \mathrm{d}\phi \\ &= \int_0^{2\pi}\int_{-1}^{1} \left((a^2c^2 + (b^2 - a^2)c^2 \cos^2\phi)(1 - \xi^2) + a^2b^2\xi^2\right)^{1/2} \mathrm{d}\xi\, \mathrm{d}\phi, \end{aligned} \tag{15}$$

with

$$\xi = \cos\,\theta. \tag{16}$$

This reduces to

$$A = 8ac\int_0^1 \sqrt{1 + \left(\left(\frac{b}{c}\right)^2 - 1\right)\xi^2}\, E\left(\left(\frac{\left(1-(b/a)^2\right)(1-\xi^2)}{1+\left((b/c)^2-1\right)\xi^2}\right)^{1/2}\right) \mathrm{d}\xi. \tag{17}$$

The argument used in the Introduction guarantees that this integral is invariant under a reorientation of the coordinate frame, i.e., under each of the six possible permutations of the triplet (a, b, c) (on using (6) once the modulus of E becomes imaginary); for instance, exchanging a and c, one finds alternatively

$$A = 8ca\int_0^1 \sqrt{1 + \left(\left(\frac{b}{a}\right)^2 - 1\right)\xi^2}\, E\left(\left(\frac{\left(1-(b/c)^2\right)(1-\xi^2)}{1+\left((b/a)^2-1\right)\xi^2}\right)^{1/2}\right) \mathrm{d}\xi. \tag{18}$$

The value of the integral in (17) (or (18)) can in fact be obtained explicitly (see the end of Section 2.2) by use of some general integral relation to be derived in the next subsection.

2.1. Integrals on an ellipsoid of revolution

By specializing to the case of an axisymmetric ellipsoid, a prolate spheroid (i.e., with $b = c$), the equality of (17) and (18) leads to the following integral over (part of) the modulus of the complete elliptic integral of the second kind (using $E(0) = \frac{1}{2}\pi$):

$$\int_0^1 E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \tfrac{1}{4}\pi\left(\sqrt{1-k^2} + \frac{\arcsin\,k}{k}\right), \quad 0 \leqslant k^2 \leqslant 1, \tag{19}$$

with $k^2 = 1 - (c/a)^2$. The range of validity of k^2 can be extended to arbitrary negative values by considering subsequently the case of an oblate spheroid (i.e., with $b = a$). In case $k = 1$, the right-hand side of (19) is $\frac{1}{8}\pi^2$, which verifies a well-known result [3, p.637].

A companion result for the complete elliptic integral of the first kind follows by choosing $F(x, y, z)$ equal to

$$F_k(x, y, z) = \frac{1-(z/c)^2}{|\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi|^2}, \tag{20}$$

whose θ, ϕ-dependence, as in D, is independent to a reorientation of the coordinate frame. The resulting integral has an argument which is the reciprocal of that in (15):

$$I = \int_0^{2\pi}\int_{-1}^1 \left((a^2c^2 + (b^2-a^2)c^2\cos^2\phi)(1-\xi^2) + a^2b^2\xi^2\right)^{-1/2} \mathrm{d}\xi\,\mathrm{d}\phi, \tag{21}$$

which can be expressed as

$$I = \frac{8}{ac}\int_0^1 \frac{1}{\sqrt{1+\left((b/c)^2-1\right)\xi^2}}\, K\left(\left(\frac{\left(1-(b/a)^2\right)(1-\xi^2)}{1+\left((b/c)^2-1\right)\xi^2}\right)^{1/2}\right) \mathrm{d}\xi, \tag{22}$$

where $K(\kappa)$ denotes the complete elliptic integral of the second kind:

$$K(\kappa)=\int_0^{\pi/2}\frac{1}{\sqrt{1-\kappa^2\sin^2\phi}}\,\mathrm{d}\phi.$$

In the same vein, integral I in (22) is equal to each of its aliases under a permutation of (a, b, c). In particular, when we choose $b=c$ and use $K(0)=\frac{1}{2}\pi$, we find

$$\int_0^1 K\left(k\sqrt{1-\xi^2}\right)\,\mathrm{d}\xi=\tfrac{1}{2}\pi\frac{\arcsin k}{k}, \tag{23}$$

which, for $k=1$, yields $\frac{1}{4}\pi^2$, in accordance with [3, p.637].

A particularly fruitful choice of the arbitrary function $F(x, y, z)$ is to take it merely a function of z, or, preferably, of z/c $(=Z(\theta)=\cos\theta=\xi)$:

$$F(x, y, z)=f(Z(\theta))=f(\xi).$$

This choice then allows the integration over ϕ to be performed, or better, to be absorbed in the definition of a fixed kernel; the complete elliptic integrals below. The functional dependence in the rotated frame then reads

$$F(z', y', x')=f(X(\theta, \phi))=f\left(\sqrt{1-\xi^2}\cos\phi\right).$$

We thus find, by the same argument, again on a surface of revolution,

$$\int_0^1 f(\xi)E\left(k\sqrt{1-\xi^2}\right)\,\mathrm{d}\xi=\int_0^1\sqrt{1-k^2\xi^2}\int_0^{\pi/2}f\left(\sqrt{1-\xi^2}\cos\phi\right)\,\mathrm{d}\phi\,\mathrm{d}\xi. \tag{24}$$

Taking similarly

$$F(x, y, z)=f(\xi)F_k(x, y, z),$$

with F_k defined in (20), yields the companion relation

$$\int_0^1 f(\xi)K\left(k\sqrt{1-\xi^2}\right)\,\mathrm{d}\xi=\int_0^1\frac{1}{\sqrt{1-k^2\xi^2}}\int_0^{\pi/2}f\left(\sqrt{1-\xi^2}\cos\phi\right)\,\mathrm{d}\phi\,\mathrm{d}\xi. \tag{25}$$

The double integral at the right-hand side of (24) and (25) as it stands has no obvious advantage over that on the left-hand side; yet, for a large class of functions, in particular polynomials, it can readily be integrated, thereby yielding what may be called the "complete elliptic integral transform" (of the second and first kind, respectively) of these functions.

A few special cases will be discussed now.

Polynomials: $f(\xi)=\xi^n$, $n=0, 1, 2,\ldots$

The first members of this class ($n=0$) have been established in the introduction of Section 2.1, viz. (19) and (23). With (24) one obtains

$$\begin{aligned}\int_0^1 \xi^n E\left(k\sqrt{1-\xi^2}\right)\,\mathrm{d}\xi&=\int_0^{\pi/2}\cos^n\phi\,\mathrm{d}\phi\int_0^1\sqrt{(1-k^2\xi^2)}\,(1-\xi^2)^{n/2}\,\mathrm{d}\xi\\&=\tfrac{1}{2}\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(n+1)\right)}{\Gamma\left(\frac{1}{2}n+1\right)}E_{n+1}(k),\end{aligned} \tag{26}$$

with $\Gamma(x)$ the Gamma function and introducing the definition (setting $\xi = \sin\tau$)

$$E_n(k) \equiv \int_0^{\pi/2} \sqrt{1-k^2\sin^2\tau}\cos^n\tau\,\mathrm{d}\tau. \tag{27}$$

The $E_n(k)$ are well known and have the following values for $n = 0, 1, 2, 3$ [8, p.202]:

$$\begin{aligned} &E_0(k) = E(k), \qquad E_1(k) = \tfrac{1}{2}\left(\sqrt{1-k^2} + \frac{\arcsin k}{k}\right),\\ &E_2(k) = \frac{1}{3k^2}\left((k^2+1)E(k) - (1-k^2)K(k)\right),\\ &E_3(k) = \frac{1}{8k^2}\left((2k^2+1)\sqrt{1-k^2} + (4k^2-1)\frac{\arcsin k}{k}\right). \end{aligned} \tag{28}$$

For $n > 3$ one may use the recursion relation

$$E_n(k) = \frac{1}{(n+1)k^2}\left((nk^2 - (n-2)(1-k^2))E_{n-2}(k) + (n-3)(1-k^2)E_{n-4}(k)\right). \tag{29}$$

Similarly, with (25) one obtains

$$\begin{aligned} \int_0^1 \xi^n K\left(k\sqrt{1-\xi^2}\right)\mathrm{d}\xi &= \int_0^{\pi/2}\cos^n\phi\,\mathrm{d}\phi \int_0^1 \frac{1}{\sqrt{(1-k^2\xi^2)}}(1-\xi^2)^{n/2}\,\mathrm{d}\xi\\ &= \frac{\frac{1}{2}\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(n+1)\right)}{\Gamma\left(\frac{1}{2}n+1\right)}K_{n+1}(k), \end{aligned} \tag{30}$$

introducing the definition

$$K_n(k) \equiv \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\tau}}\cos^n\tau\,\mathrm{d}\tau. \tag{31}$$

The $K_n(k)$ are also well known [8, p.203] and have the following values for $n = 0, 1, 2, 3$:

$$\begin{aligned} &K_0(k) = K(k), \qquad K_1(k) = \frac{\arcsin k}{k},\\ &K_2(k) = \frac{1}{k^2}\left(E(k) - (1-k^2)K(k)\right),\\ &K_3(k) = \frac{1}{2k^2}\left(\sqrt{1-k^2} + (2k^2-1)\frac{\arcsin k}{k}\right), \end{aligned} \tag{32}$$

while the recursion relation in this case reads

$$K_n(k) = \frac{1}{(n-1)k^2}\left((n-2)(2k^2-1)K_{n-2}(k) + (n-3)(1-k^2)K_{n-4}(k)\right). \tag{33}$$

By substituting

$$\xi = \sqrt{1-\left(\frac{\mu}{k}\right)^2} \tag{34}$$

in the left-hand side of (26), this integral can, for odd powers $n = 2m + 1$, $m = 0, 1, 2, \ldots,$ be rewritten as

$$\int_0^1 \xi^{2m+1} E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \frac{1}{k^2}\int_0^k \left(1 - \left(\frac{\mu}{k}\right)^2\right)^m \mu E(\mu)\,\mathrm{d}\mu. \tag{35}$$

This is, for integer values of m, related to tabulated integrals with varying upper limit of integration, since the integrand is recognized as a combination of odd polynomials in μ multiplied by $E(\mu)$ [3, p.627]. Thus, this merely verifies an existing result. A similar remark is valid for the companion integral with the complete elliptic integral of the first kind as kernel. The complementary class, of even-powered polynomials, however, is missing in tables of integration.

Integral relationships: $f(\xi) = (1 - \xi^2)^{\pm 1/2}$

A remarkable class of identities among integrals over complete elliptic integrals results by considering

$$f(\xi) = \frac{1}{\sqrt{1-\xi^2}}.$$

By (24),

$$\int_0^1 \frac{1}{\sqrt{1-\xi^2}} E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \int_0^1 \sqrt{1-k^2\xi^2} \int_0^{\pi/2} \frac{1}{\sqrt{1-(1-\xi^2)\cos^2\phi}} \mathrm{d}\phi\,\mathrm{d}\xi$$
$$= \int_0^1 \sqrt{1-k^2\xi^2}\,\frac{1}{\xi} K\left(\mathrm{i}\frac{\sqrt{1-\xi^2}}{\xi}\right) \mathrm{d}\xi.$$

With the use of

$$K(\mathrm{i}k) = \frac{1}{\sqrt{1+k^2}} K\left(\frac{k}{\sqrt{1+k^2}}\right),$$

the equivalent of (6), this reduces to

$$\int_0^1 \frac{1}{\sqrt{1-\xi^2}} E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \int_0^1 \sqrt{1-k^2\xi^2}\, K\left(\sqrt{1-\xi^2}\right) \mathrm{d}\xi. \tag{36}$$

With the same $f(\xi)$, (25) yields

$$\int_0^1 \frac{1}{\sqrt{1-\xi^2}} K\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \int_0^1 \frac{1}{\sqrt{1-k^2\xi^2}} K\left(\sqrt{1-\xi^2}\right) \mathrm{d}\xi. \tag{37}$$

These two relations are complementary to another set obtained by choosing

$$f(\xi) = \sqrt{1-\xi^2}.$$

Inserting this in (24) and (25) leads to

$$\int_0^1 \sqrt{1-\xi^2}\, E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \int_0^1 \sqrt{1-k^2\xi^2}\, E\left(\sqrt{1-\xi^2}\right) \mathrm{d}\xi, \tag{38}$$

$$\int_0^1 \sqrt{1-\xi^2}\, K\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \int_0^1 \frac{1}{\sqrt{1-k^2\xi^2}} E\left(\sqrt{1-\xi^2}\right) \mathrm{d}\xi. \tag{39}$$

No explicit evaluation of the integrals above is obtained, however.

Relations following by differentiation

Using the rules for differentiation of complete elliptic integrals

$$\frac{\mathrm{d}}{\mathrm{d}k} K(k) = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}, \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}k} E(k) = \frac{E(k) - K(k)}{k}, \tag{41}$$

some other integrals can be derived from those previously obtained. By, for instance, taking the partial k-derivative of (24), one obtains, by employing (41),

$$\int_0^1 f(\xi) K\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \left(1 - k\frac{\partial}{\partial k}\right) \int_0^1 f(\xi) E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi. \tag{42}$$

One may check that this again leads to (25), when (24) is used to replace the second integral in (42). Taking the partial k-derivative of (25), using (40), leads to

$$\int_0^1 \frac{f(\xi)}{1-k^2(1-\xi^2)} E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \left(1 + k\frac{\partial}{\partial k}\right) \int_0^1 f(\xi) K\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi, \tag{43}$$

which, with (42), reads

$$\int_0^1 \frac{f(\xi)}{1-k^2(1-\xi^2)} E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi = \left(1 - k\frac{\partial}{\partial k} - k^2\frac{\partial^2}{\partial k^2}\right) \int_0^1 f(\xi) E\left(k\sqrt{1-\xi^2}\right) \mathrm{d}\xi. \tag{44}$$

Therefore, once a particular integral, with $E(k\sqrt{1-\xi^2})$ as kernel, is found, (44) shows that a companion integral is obtained by differentiation.

2.2. *Integrals on an arbitrary ellipsoid*

Eq. (24), derived for arbitrary functions $f(\xi)$, can be successfully applied to the general case in which integrals over an ellipsoid, having semi-axes a, b and c of different length, are studied. In the following we will first restrict ourselves to the unit field $F(x, y, z) = 1$, which will lead to an explicit expression for the surface area of an arbitrary ellipsoid. The extension to other fields will then be obvious.

The surface area of an ellipsoid, given by (17),

$$A = 8ac\int_0^1 \sqrt{1+\mu^2\xi^2}\, E\left(\left(\frac{\nu^2(1-\xi^2)}{1+\mu^2\xi^2}\right)^{1/2}\right) \mathrm{d}\xi, \tag{45}$$

where

$$\mu^2 = \left(\frac{b}{c}\right)^2 - 1, \qquad \nu^2 = 1 - \left(\frac{b}{a}\right)^2, \tag{46}$$

can be brought in the "canonical" form of (24) by the transformation

$$\frac{1-\xi^2}{1+\mu^2\xi^2} = 1 - \eta^2, \tag{47}$$

which choice guarantees that the integration boundaries remain unchanged. Inserting

$$\xi^2 = \frac{\eta^2}{1+\mu^2(1-\eta^2)},$$

obtained from (47), into (45) yields

$$A = \frac{8ac}{\sqrt{1+\mu^2}} \int_0^1 \frac{1}{\left(1-\mu^2\eta^2/(1+\mu^2)\right)^2} E\left(\nu\sqrt{1-\eta^2}\right) \mathrm{d}\eta. \tag{48}$$

Applying (24), this is equal to

$$A = \frac{8ac}{\sqrt{1+\mu^2}} \int_0^1 \sqrt{1-\nu^2\eta^2} \int_0^{\pi/2} \frac{1}{\left(1-\mu^2/(1+\mu^2)(1-\eta^2)\cos^2\phi\right)^2} \mathrm{d}\phi\, \mathrm{d}\eta. \tag{49}$$

With

$$\int_0^{\pi/2} \frac{1}{\left(1+\rho\cos^2\phi\right)^2} \mathrm{d}\phi = \tfrac{1}{4}\pi \frac{2+\rho}{(1+\rho)^{3/2}}$$

(cf. [8, p.182]) and rewriting the remaining integral, the surface area of the ellipsoid (on substituting from (46)) can be obtained in terms of special functions:

$$A = 2\pi ac\left(\frac{c}{a} + \frac{b}{a}\sqrt{1-\left(\frac{c}{a}\right)^2}\, E(\varphi, \kappa) + \frac{b}{a}\frac{c}{a}\frac{1}{\sqrt{1-(c/a)^2}} F(\varphi, \kappa)\right), \tag{50a}$$

with

$$\varphi = \arcsin\sqrt{1-\left(\frac{c}{a}\right)^2}, \qquad \kappa^2 = \frac{1-(c/b)^2}{1-(c/a)^2}. \tag{50b}$$

Here, $F(\varphi, \kappa)$ and $E(\varphi, \kappa)$ denote the (incomplete) elliptic integrals of the first and second kind, respectively. The explicit expression for the surface area of an ellipsoid (50) agrees with the classical result [7], which, however, appears to have sunk into oblivion.

By comparing (45), (48) and (50a), this can also be read as the two-dimensional "elliptic integral transform" of the unit function:

$$\int_0^1 \sqrt{1+\mu^2\xi^2}\, E\left(\left(\frac{\nu^2(1-\xi^2)}{1+\mu^2\xi^2}\right)^{1/2}\right) \mathrm{d}\xi$$

$$= \frac{1}{\sqrt{1+\mu^2}} \int_0^1 \frac{1}{\left(1-\mu^2\eta^2/(1+\mu^2)\right)^2} E\left(\nu\sqrt{1-\eta^2}\right) \mathrm{d}\eta$$

$$= \tfrac{1}{4}\pi\left(\left(\frac{1-\nu^2}{1+\mu^2}\right)^{1/2} + \sqrt{\nu^2+\mu^2}\, E(\varphi, \kappa) + \frac{1-\nu^2}{\sqrt{\nu^2+\mu^2}} F(\varphi, \kappa)\right), \tag{51a}$$

with

$$\varphi = \arcsin\left(\frac{\nu^2+\mu^2}{1+\mu^2}\right)^{1/2}, \qquad \kappa^2 = \frac{\mu^2}{\nu^2+\mu^2}. \tag{51b}$$

By again choosing $F(x, y, z) = f(\xi)$, one obtains, with (47), more generally

$$\int_0^1 f(\xi)\sqrt{1+\mu^2\xi^2}\, E\left(\left(\frac{\nu^2(1-\xi^2)}{1+\mu^2\xi^2}\right)^{1/2}\right) \mathrm{d}\xi$$

$$= \frac{1}{\sqrt{1+\mu^2}} \int_0^1 \frac{f\left(\eta/\sqrt{1+\mu^2(1-\eta^2)}\right)}{\left(1-\mu^2\eta^2/(1+\mu^2)\right)^2} E\left(\nu\sqrt{1-\eta^2}\right) \mathrm{d}\eta$$

$$= \frac{1}{\sqrt{1+\mu^2}} \int_0^1 \sqrt{1-\nu^2\eta^2} \int_0^{\pi/2} \frac{f\left(\sqrt{1-\eta^2}\cos\phi/\sqrt{1+\mu^2\left(1-(1-\eta^2)\cos^2\phi\right)}\right)}{\left(1-\mu^2/(1+\mu^2)(1-\eta^2)\cos^2\phi\right)^2} \mathrm{d}\phi\, \mathrm{d}\eta, \tag{52}$$

which, conceivably, is, for some functions $f(\xi)$, more readily integrated than its left-hand side.

By the same substitution (47), one may now also evaluate the integral I in (22):

$$I = \frac{8}{ac}\int_0^1 \frac{1}{\sqrt{1+\mu^2\xi^2}} K\left(\left(\frac{\nu^2(1-\xi^2)}{1+\mu^2\xi^2}\right)^{1/2}\right) \mathrm{d}\xi = \frac{4\pi}{ac}\frac{1}{\sqrt{\nu^2+\mu^2}} F(\varphi, \kappa). \tag{53}$$

From this we find similarly

$$\int_0^1 \frac{1}{\sqrt{1+\mu^2\xi^2}} K\left(\left(\frac{\nu^2(1-\xi^2)}{1+\mu^2\xi^2}\right)^{1/2}\right) \mathrm{d}\xi = \frac{1}{\sqrt{1+\mu^2}} \int_0^1 \frac{1}{1-\mu^2\eta^2/(1+\mu^2)} K\left(\nu\sqrt{1-\eta^2}\right) \mathrm{d}\eta$$

$$= \tfrac{1}{2}\pi \frac{1}{\sqrt{\nu^2+\mu^2}} F(\varphi, \kappa), \tag{54}$$

with φ and κ as in (51b). A generalization of this result, for arbitrary $f(\xi)$, can, as in (52), readily be constructed along the same lines and will not be given explicitly here.

The list of specific functions $f(\xi)$, treated in this section, is by no means exhaustive and further study is needed to elucidate its entire range.

3. Discussion and conclusions

A number of possible generalizations are conceived, which, however, have not been pursued here and which could be a valuable subject for future research.

(1) The two cases treated in the previous section may be characterized as evaluations of

$$\int_0^{2\pi}\int_{-1}^{1}\left((a^2c^2+(b^2-a^2)c^2\cos^2\phi)(1-\xi^2)+a^2b^2\xi^2\right)^{\lambda}\,\mathrm{d}\xi\,\mathrm{d}\phi,$$

for $\lambda = \pm\frac{1}{2}$, see (15) and (21). It will be interesting to verify whether other choices of λ lead to additional results.

(2) Can one adopt other functions of $(a^2c^2+(b^2-a^2)c^2\cos^2\phi)(1-\xi^2)+2a^2b^2\xi^2$ to act as a kernel in an integral, while retaining some of the features in the previous analysis?

(3) The integrations have been performed on a "stretched sphere", the ellipsoid. This is a quadric, which is quadratic in each of its variables. The symmetric appearance of the three coordinates seems to be the basis for the possible exchange of the "longitudinal" x-axis and the "co-latitudinal" z-axis and its subsequent alternative evaluation of an integral. Such an exchange is apparently missing in most other (generalized) coordinate systems, as the z- and, with it, θ-dependence is usually more involved than the x-, y- and, with it, ϕ-dependence [6]. Retaining the symmetry in each of the coordinates, however, still allows one to consider integrals of fields on surfaces determined by

$$\left(\frac{x}{a}\right)^{2\lambda}+\left(\frac{y}{b}\right)^{2\lambda}+\left(\frac{z}{c}\right)^{2\lambda}=1.$$

It is intriguing to check whether an evaluation of these fields from different perspectives may lead to additional results for values of λ other than 1.

(4) It was observed that the "change of perspective" in the two-dimensional case is, besides that it enables the derivation of an identity like (6), not very fruitful; see the Introduction. Three-dimensional space, in which the "longitudinal" and "co-latitudinal" axes can be exchanged, seems to be much richer in that respect. It is thus natural to expect that additional results can be obtained from the evaluation (from different perspectives) of integrals of fields on $(n-1)$-dimensional, symmetrical surfaces in n-dimensional space (where $n>3$).

Even though the emphasis in this paper is on the evaluation of certain integrals of elliptic integrals, the results are, of course, of direct interest to applied fields. This study, for once, was initiated by the question whether the growth of some plankton species (tentatively modelled as ellipsoids) is either dominated by internal (α volume), or by flux-related, external (α surface area) processes. By observing whether their mass increase in time is proportional to either their volume, or surface area, this question is presumably resolved. Other aspects of such animal life, like the pressure forces executed upon them by the ambient fluid, or their temperature, will likewise require the evaluation of certain fields over their surface. Other applications may arise in geophysics, when evaluating, e.g., the angular momentum of the atmosphere, at the surface of a (rotationally modified) planetary ellipsoid, or, in a purely mechanical context, when determining, e.g., the moment of inertia of a void ellipsoid (rugby ball), where all the mass is concentrated in the hull.

In conclusion, the evaluation of the integral value of fields on an ellipsoid can be done from different perspectives, which lead to different expressions. The fact that this integral value should be independent from the adopted frame of reference assures that these expressions should be identical. In this way fairly general identities (24) and (25) are formulated, in which, on the one hand, the kernel is formed by a complete elliptic integral, and, on the other hand by a radical (in which case, however, one additional integration has to be performed). An application of this "elliptic integral transform" is made in the calculation of the surface area of an arbitrarily shaped ellipsoid, since the defining integral can, by a proper transformation, be made conforming to the requirements in (24). The equivalent integral expression, which this equation describes, can be integrated and the surface area of the ellipsoid can be rederived explicitly in terms of (incomplete) elliptic integrals (50).

Note. After completion of this paper, the author, through a reference in [4], discovered that the original derivation of the surface area of an ellipsoid has been given by Legendre [5]. A more symmetrized approach of elliptic integrals — symmetry aspects which lie at the heart of the results presented here — and contemporary derivation of the surface area of the ellipsoid can be found in [2].

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